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Spaces of maps from the closed Riemann surface into the 2-sphere

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1 Introduction.

For connected spaces X and Y , let $\text{Map}(X, Y)$ (resp. $\text{Map}^*(X, Y)$) denote the space consisting of all continuous (resp. based continuous) maps $f : X \rightarrow Y$ with compact-open topology. Let T_g denote the closed Riemann surface of genus g . Then for each integer $d \in \mathbb{Z} = \pi_0(\text{Map}(T_g, S^2))$ we denote by $\text{Map}_d(T_g, S^2)$ (resp. by $\text{Map}_d^*(T_g, S^2)$) the corresponding path-component of $\text{Map}(T_g, S^2)$ (resp. $\text{Map}^*(T_g, S^2)$) consisting of all maps (resp. of base-point preserving maps) $f : T_g \rightarrow S^2$ of degree d . Similarly, we denote by $\text{Hol}_d(T_g, S^2)$ the subspace of $\text{Map}_d(T_g, S^2)$ of all holomorphic maps $f : T_g \rightarrow S^2$ of degree d , and by $\text{Hol}_d^*(T_g, S^2)$ the corresponding subspace of $\text{Map}_d^*(T_g, S^2)$ of all base-point preserving holomorphic maps of degree d . Note that $\text{Hol}_d(T_g, S^2) = \emptyset$ if $d < 0$ and that any holomorphic map $f : T_g \rightarrow S^2$ of degree zero is a constant map. So in this paper we always assume that $d \geq 1$ and recall the following results.

Theorem 1.1 (L. Larmore and E. Thomas, [7]). (i) *If $g = 0$, $T_0 = S^2$ and there are isomorphisms*

$$\pi_1(\text{Map}_d(S^2, S^2)) \cong \mathbb{Z}/2d, \quad \pi_1(\text{Map}_d^*(S^2, S^2)) \cong \mathbb{Z}.$$

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(ii) If $g \geq 1$, there are isomorphisms

$$\begin{cases} \pi_1(\mathrm{Map}_d(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2, \alpha^{2d} = 1 \rangle, \\ \pi_1(\mathrm{Map}_d^*(T_g, S^2)) \cong \langle \alpha, e_j \ (1 \leq j \leq 2g) | [e_k, e_{k+g}] = \alpha^2 \rangle, \end{cases}$$

where $k = 1, 2, \dots, g$, and $[x, y] = xyx^{-1}y^{-1}$. \square

Theorem 1.2 (G. Segal, [9]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(T_g, S^2) \rightarrow \mathrm{Map}_d^*(T_g, S^2) \\ j_d : \mathrm{Hol}_d(T_g, S^2) \rightarrow \mathrm{Map}_d(T_g, S^2) \end{cases}$$

are homotopy equivalences up to dimension d if $g = 0$, and they are homology equivalences up to dimension $D(d; g) = d - 2g$ if $g \geq 1$. \square

Theorem 1.3 (S. Kallel, [6]). *If $d > 2g$ and $g \geq 1$, the inclusion maps i_d and j_d induce isomorphisms*

$$\begin{cases} i_{d*} : \pi_1(\mathrm{Hol}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*(T_g, S^2)), \\ j_{d*} : \pi_1(\mathrm{Hol}_d(T_g, S^2)) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d(T_g, S^2)). \end{cases} \quad \square$$

Remark. A map $f : X \rightarrow Y$ is called a *homotopy (resp. homology) equivalence up to dimension D* if $f_* : \pi_k(X) \rightarrow \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and epimorphism for $k = D$.

We expect that the inclusions i_d and j_d will be homotopy equivalences up to dimension $D(d; g)$ for $g \geq 1$. For example, Theorem 1.3 supports that this might be true, and it seems valuable to investigate the homotopy types of the universal coverings of $\mathrm{Hol}_d^*(T_g, S^2)$ and $\mathrm{Hol}_d(T_g, S^2)$.

The main purpose of this note is to announce the recent work of the author given in [13], in which we shall study the homotopy types of universal coverings of the above spaces. Let \tilde{X} denote the universal covering of a connected space X . Then we can state our results as follows.

Theorem 1.4 ([13]). *If $d \geq 1$, there is a homotopy equivalence*

$$\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}}_d^*(T_g, S^2) \xrightarrow{\cong} \widetilde{\mathrm{Hol}}_d(T_g, S^2).$$

Corollary 1.5 ([1], [8], [12]). *If $g = 0$ and $d \geq 1$, there is a homotopy equivalence $\tilde{\Phi}_d : S^3 \times \widetilde{\mathrm{Hol}}_d^*(S^2, S^2) \xrightarrow{\cong} \widetilde{\mathrm{Hol}}_d(S^2, S^2)$.* \square

Let $i : \text{Hol}_d^*(T_g, S^2) \rightarrow \text{Hol}_d(T_g, S^2)$ be an inclusion map and let $ev : \text{Hol}_d(T_g, S^2) \rightarrow S^2$ denote the evaluation map given by $ev(f) = f(t_0)$, where $t_0 \in T_g$ is the base-point of T_g . Then it is known that there is a evaluation fibration sequence (e.g. [6])

$$\text{Hol}_d^*(T_g, S^2) \xrightarrow[\subset]{i} \text{Hol}_d(T_g, S^2) \xrightarrow{ev} S^2.$$

Corollary 1.6 ([13]). *If $k \geq 2$ and $d \geq 1$, the above sequence induces a split short exact sequence*

$$0 \rightarrow \pi_k(\text{Hol}_d^*(T_g, S^2)) \xrightarrow{i_*} \pi_k(\text{Hol}_d(T_g, S^2)) \xrightarrow{ev_*} \pi_k(S^2) \rightarrow 0.$$

Theorem 1.7 ([13]). *Let $d \geq 1$ be an integer.*

(i) *There is a homotopy equivalence*

$$\tilde{\Phi} : S^3 \times \widetilde{\text{Map}}^*(T_g, S^3) \xrightarrow{\simeq} \widetilde{\text{Map}}_d(T_g, S^2)$$

and there is a fibration sequence (up to homotopy equivalence)

$$\Omega^2 S^3 \langle 3 \rangle \rightarrow \widetilde{\text{Map}}^*(T_g, S^3) \rightarrow (\Omega S^3)^{2g},$$

where $S^3 \langle 3 \rangle$ denotes the 3-connective covering of S^3 .

(ii) *For any $k \geq 2$, there is an isomorphism*

$$\pi_k(\widetilde{\text{Map}}^*(T_g, S^3)) \cong \pi_{k+2}(S^3) \oplus \pi_{k+1}(S^3)^{\oplus 2g}.$$

2 The idea of the proofs.

Let X , Y and Z be connected spaces and let $f : X \rightarrow Y$ be a map. Define the map $f^\# : \text{Map}(Y, Z) \rightarrow \text{Map}(X, Z)$ by $f^\#(g) = g \circ f$ for $g \in \text{Map}(Y, Z)$. Similarly, we define the map $f^\# : \text{Map}^*(Y, Z) \rightarrow \text{Map}^*(X, Z)$ by the restriction. It is well known that there is a cofiber sequence

$$(1) \quad S^1 \xrightarrow{\varphi_g} \vee^{2g} S^1 \xrightarrow{i'} T_g \xrightarrow{q_g} S^2 \xrightarrow{\Sigma \varphi_g} \vee^{2g} S^2$$

where $\pi_1(\vee^{2g} S^1)$ is the free group on $2g$ generators $\{a_j, b_j : 1 \leq j \leq g\}$ and $\varphi_g = [a_1, b_1][a_2, b_2] \cdots [a_g, b_g] \in \pi_1(\vee^{2g} S^1)$.

Lemma 2.1. $q_g^\# : \pi_k(\Omega_d^2 S^2) \rightarrow \pi_k(\text{Map}_d^*(T_g, S^2))$ is a monomorphism for any $k \geq 1$.

Proof. By using an easy diagram chasing we can show the assertion. \square

Consider the commutative diagram of evaluation fibration sequences

$$(2) \quad \begin{array}{ccccc} \text{Map}_d^*(S^2, S^2) & \xrightarrow[\subset]{i_T} & \text{Map}_d(S^2, S^2) & \xrightarrow{ev_S} & S^2 \\ q_g^\# \downarrow & & q_g^\# \downarrow & & \parallel \\ \text{Map}_d^*(T_g, S^2) & \xrightarrow[\subset]{i_S} & \text{Map}_d(T_g, S^2) & \xrightarrow{ev_T} & S^2 \end{array}$$

Lemma 2.2. $ev_{X*} : \pi_2(\text{Map}_d(X, S^2)) \rightarrow \pi_2(S^2)$ is trivial for $X \in \{S^2, T_g\}$.

Proof. If $X = S^2$, we can easily show the assertion by using Whitehead's Theorem [11] concerning the boundary operator of the homotopy exact sequence induced from the evaluation fibration. By using this with the diagram chasing of (2), we can also prove the assertion for $X = T_g$. \square

Lemma 2.3. The induced homomorphisms

$$\begin{cases} i_{T*} : \pi_2(\text{Map}_d^*(T_g, S^2)) \rightarrow \pi_2(\text{Map}_d(T_g, S^2)) \\ i_* : \pi_2(\text{Hol}_d^*(T_g, S^2)) \rightarrow \pi_2(\text{Hol}_d(T_g, S^2)) \end{cases}$$

are epimorphisms.

Proof. The proof follows from the diagram chasing and Lemma 2.2. \square

If we identify $S^2 = \mathbb{CP}^1$, the group $SU(2)$ acts on S^2 by the right matrix multiplication. By using this right matrix multiplication, define the map

$$(3) \quad \Phi : SU(2) \times \text{Map}_d^*(T_g, S^2) \rightarrow \text{Map}_d(T_g, S^2)$$

by $(\Phi(A, f))(t) = f(t) \cdot A$ for $(A, f, t) \in SU(2) \times \text{Map}_d^*(T_g, S^2) \times T_g$.

Since $\Phi(SU(2) \times \text{Hol}_d^*(T_g, S^2)) \subset \text{Hol}_d(T_g, S^2)$, we can define the map

$$(4) \quad \Phi_d : SU(2) \times \text{Hol}_d^*(T_g, S^2) \rightarrow \text{Hol}_d(T_g, S^2)$$

by the restriction $\Phi_d = \Phi|_{SU(2) \times \text{Hol}_d^*(T_g, S^2)}$.

Theorem 2.4 ([13]). $\Phi_{d*} : \pi_k(SU(2) \times \text{Hol}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_k(\text{Hol}_d(T_g, S^2))$ is an isomorphism for any $k \geq 2$. \square

Proof. The detail is omitted and see [13] in detail. \square

Now we can prove Theorem 1.4 by using Theorem 2.4.

Proof of Theorem 1.4. Let

$$\begin{cases} \pi : \widetilde{\text{Hol}}_d(T_g, S^2) \rightarrow \text{Hol}_d(T_g, S^2) \\ \pi' : \widetilde{\text{Hol}}_d^*(T_g, S^2) \rightarrow \text{Hol}_d^*(T_g, S^2) \end{cases}$$

denote projection maps of the universal coverings. If we identify $SU(2) = S^3$, it is easy to see that the universal covering of $SU(2) \times \text{Hol}_d^*(T_g, S^2)$ is given by $1 \times \pi' : S^3 \times \widetilde{\text{Hol}}_d^*(T_g, S^2) \rightarrow SU(2) \times \text{Hol}_d^*(T_g, S^2)$. Then because π is a projection of the universal covering, there is a lifting $\tilde{\Phi}_d : S^3 \times \widetilde{\text{Hol}}_d^*(T_g, S^2) \rightarrow \widetilde{\text{Hol}}_d(T_g, S^2)$ such that the following diagram is commutative.

$$\begin{array}{ccc} S^3 \times \widetilde{\text{Hol}}_d^*(T_g, S^2) & \xrightarrow{\tilde{\Phi}_d} & \widetilde{\text{Hol}}_d(T_g, S^2) \\ 1 \times \pi' \downarrow & & \pi \downarrow \\ SU(2) \times \text{Hol}_d^*(T_g, S^2) & \xrightarrow{\Phi_d} & \text{Hol}_d(T_g, S^2) \end{array}$$

Since $\pi_k(\Phi_d)$ is an isomorphism for any $k \geq 2$, an easy diagram chasing shows that $\pi_k(\tilde{\Phi}_d)$ is also an isomorphism for any $k \geq 2$. Because $S^3 \times \widetilde{\text{Hol}}_d^*(T_g, S^2)$ and $\widetilde{\text{Hol}}_d(T_g, S^2)$ are simply connected, $\tilde{\Phi}_d$ is a homotopy equivalence. \square

By using the completely similar way as above, we can see that there is a fibration sequence (up to homotopy equivalence)

$$(5) \quad S^1 \longrightarrow SU(2) \times \text{Map}_d^*(T_g, S^2) \xrightarrow{\Phi} \text{Map}_d(T_g, S^2).$$

Theorem 2.5 ([13]). $\Phi_* : \pi_k(SU(2) \times \text{Map}_d^*(T_g, S^2)) \xrightarrow{\cong} \pi_k(\text{Map}_d(T_g, S^2))$ is an isomorphism for any $k \geq 2$.

Proof. The proof is analogous to that of Theorem 2.4. \square

Corollary 2.6 ([13]). *There is a homotopy equivalence*

$$\widetilde{\text{Map}}_d(T_g, S^2) \simeq S^3 \times \widetilde{\text{Map}}_0^*(T_g, S^2). \quad \square$$

Lemma 2.7. *There is a homotopy fibration sequence*

$$\Omega^2 S^3 \langle 3 \rangle \rightarrow \widetilde{\text{Map}}^*(T_g, S^3) \rightarrow (\Omega S^3)^{2g}.$$

Proof. This can be proved by using tedious diagram chasing and the detail is omitted. \square

Proof of Theorem 1.7. (i) By using Corollary 2.6 and Lemma 2.7, to prove (i) it is sufficient to show that there is a homotopy equivalence

$$(6) \quad \widetilde{\text{Map}}^*(T_g, S^3) \simeq \widetilde{\text{Map}}_0^*(T_g, S^2).$$

However, by using a similar manner as that of the previous Theorem we can prove the assertion (i) and the detail is omitted.

(ii) Assume that $k \geq 2$. Since there is a homotopy equivalence $\Sigma^k T_g \simeq \vee^{2g} S^{1+k} \vee S^{2+k}$ and S^3 is a Lie group, $\text{Map}^*(T_g, S^3)$ is an H-space. Hence, the assertion (ii) easily follows. \square

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